A Survey of Fully Homomorphic Encryption

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Homomorphic Encryption

Homomorphic encryption: perform operations on plaintexts while manipulating only ciphertexts.

- Normally, this is not possible.

$$\begin{aligned}
\mathrm{AES}_K(m_1) &= \texttt{0x3c7317c6bc5634a4ad8479c64714f4f8} \\
\mathrm{AES}_K(m_2) &= \texttt{0x7619884e1961b051be1aa407da6cac2c} \\
\mathrm{AES}_K(m_1 \quad m_2) &= \;?
\end{aligned}$$

For some cryptosystems with algebraic structure, this is possible. For example RSA:

$$\left.\begin{aligned} c_1 &= {m_1}^e \bmod N \\ c_2 &= {m_2}^e \bmod N \end{aligned}\right) \quad c_1 \quad c_2 = (m_1 \quad m_2)^e \bmod N$$

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Homomorphic Encryption with RSA

Multiplicative property of RSA.

$$\left.\begin{array}{l} c_1 = {m_1}^e \bmod N \\ c_2 = {m_2}^e \bmod N \end{array}\right\} \Rightarrow c = c_1 \cdot c_2 = (m_1 \cdot m_2)^e \bmod N$$

Homomorphic encryption: given c_1 and c_2, we can compute the ciphertext c for $m_1 \cdot m_2 \bmod N$

- using only the public-key
- without knowing the plaintexts m_1 and m_2.

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Paillier Cryptosystem

Additively homomorphic: Paillier cryptosystem

$$\left.\begin{array}{l} c_1 = g^{m_1} \bmod N^2 \\ c_2 = g^{m_2} \bmod N^2 \end{array}\right. \Rightarrow c_1 \cdot c_2 = g^{m_1 + m_2\,[N]} \bmod N^2$$

Application: e-voting.

- Voter i encrypts his vote $m_i \in \{0,1\}$ into:

$$c_i = g^{m_i} \cdot z_i^N \bmod N^2$$

- Votes can be aggregated using only the public-key:

$$c = \prod_i c_i = g^{\sum_i m_i} \cdot z \bmod N^2$$

- c is enventually decrypted to recover $m = \sum_i m_i$

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- Open problem until Gentry's breakthrough in 2009.

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Fully homomorphic public-key encryption

We restrict ourselves to public-key encryption of a single bit:

- $0 \; ! \; 203ef6124\ldots23ab87_{16}$
- $1 \; ! \; b327653c1\ldots db3265_{16}$
- Obviously, encryption must be probabilistic.

Fully homomorphic property

- Given $E(b_0)$ and $E(b_1)$, one can compute $E(b_0 \quad b_1)$ and $E(b_0 \quad b_1)$ without knowing the private-key.

Why is it important ?

- Universality: any Boolean circuit can be written with Xors and Ands.
- Once you can homomorphically evaluate both a Xor and a And, you can evaluate any Boolean circuit, any computable function.

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Outsourcing Computation

The cloud receives some data m in encrypted form.

- It receives the ciphertexts c_i corresponding to bits m_i
- The cloud doesn't know the m_i's

The cloud performs some computation $f(m)$, but without knowing m

- The computation of f is written as a Boolean circuit with Xors and Ands
- Every Xor $z = x \quad y$ is homomorphically evaluated from the ciphertexts c_x and c_y, to get ciphertext c_z
- Every And $z' = x \quad y$ is homomorphically evaluated from the ciphertexts c_x and c_y, to get ciphertext $c_{z'}$

Eventally the cloud obtains a ciphertext c for $f(m)$

- The user decrypts c to recover $f(m)$
- The cloud learns nothing about m

What fully homomorphic encryption brings you

You have a software that given the revenue, past income, headcount, etc., of a company can predict its future stock price.

- I want to know the future stock price of my company, but I don't want to disclose con dential information.
- And you don't want to give me your software containing secret formulas.

Using homomorphic encryption:

- I encrypt all the inputs using fully homomorphic encryption and send them to you in encrypted form.
- You process all my inputs, viewing your software as a circuit.
- You send me the result, still encrypted.
- I decrypt the result and get the predicted stock price.
- You didn't learn any information about my company.

More generally:

- Cool buzzwords like secure cloud computing.
- Cool mathematical challenges.

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Cloud Computing

Goal: cloud computing

- I encrypt my data before sending it to the cloud
- The cloud can still search, sort and edit my data on my behalf
- Data is kept in encrypted form in the cloud.
- The cloud learns nothing about my data

The cloud returns encrypted answers

- that only I can decrypt

Fully Homomorphic Encryption Schemes

1. Breakthrough scheme of Gentry [G09], based on ideal lattices. Some optimizations by [SV10].
   - Implementation [GH11]: PK size: 2.3 GB, recrypt: 30 min.
2. RLWE schemes [BV11a,BV11b].
   - FHE without bootstrapping (modulus switching) [BGV11]
   - Batch FHE [GHS12]
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   - And many other papers...
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The DGHV Scheme

Ciphertext for $m \in \{0,1\}$:

$$c = q \cdot p + 2r + m$$

where p is the secret-key, q and r are randoms.

Decryption:

$$(c \bmod p) \bmod 2 = m$$

Parameters:

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Homomorphic Properties of DGHV

Addition:

$$\begin{array}{l} c_1 = q_1 \quad p + 2r_1 + m_1 \\ c_2 = q_2 \quad p + 2r_2 + m_2 \end{array} \;)\; c_1 + c_2 = q' \quad p + 2r' + m_1 + m_2$$

- $c_1 + c_2$ is an encryption of $m_1 + m_2 \bmod 2 = m_1 \quad m_2$

Multiplication:

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with

$$r'' = 2r_1r_2 + r_1m_2 + r_2m_1$$

- $c_1 \quad c_2$ is an encryption of $m_1 \quad m_2$
- Noise becomes twice larger.

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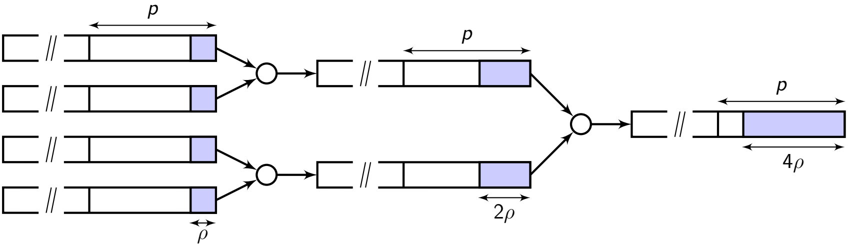
Somewhat homomorphic scheme

The number of multiplications is limited.

- Noise grows with the number of multiplications.
- Noise must remain $< p$ for correct decryption.



Gentry's technique

To build a FHE scheme, start from the somewhat homomorphic scheme, that is:

- Only a polynomial of small degree can be homomorphically applied on ciphertexts.
- Otherwise the noise becomes too large and decryption becomes incorrect.

Then, \squash" the decryption procedure:

- express the decryption function as a low degree polynomial in the bits of the ciphertext c and the secret key sk (equivalently a boolean circuit of small depth).

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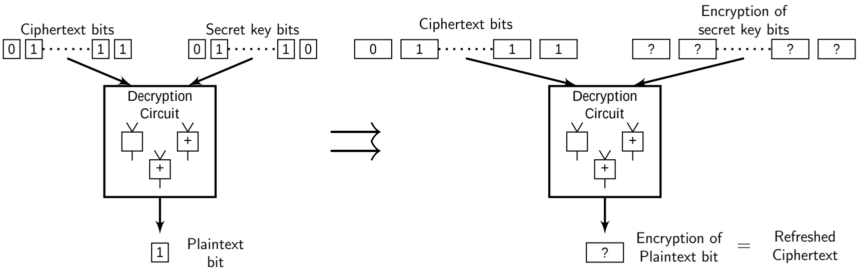
Ciphertext refresh: bootstrapping

Gentry's breakthrough idea: refresh the ciphertext using the decryption circuit homomorphically.

- Evaluate the decryption polynomial not on the bits of the ciphertext c and the secret key sk, but homomorphically on the encryption of those bits.
- Instead of recovering the bit plaintext m, one gets an encryption of this bit plaintext, *i.e.* yet another ciphertext for the same plaintext.



Ciphertext refresh

Refreshed ciphertext:

- If the degree of the decryption polynomial is small enough, the resulting noise in this new ciphertext can be smaller than in the original ciphertext

Fully homomorphic encryption:

- Given two refreshed ciphertexts one can apply again the homomorphic operation (either addition or multiplication), which was not necessarily possible on the original ciphertexts because of the noise threshold.
- Using this \ciphertext refresh" procedure the number of homomorphic operations becomes unlimited and we get a fully homomorphic encryption scheme.

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Public-key Encryption with DGHV

Ciphertext

$$c = q \cdot p + 2r + m$$

Public-key: a set of τ encryptions of 0's.

$$x_i = q_i \cdot p + 2r_i$$

Public-key encryption:

$$c = m + 2r + \sum_{i=1}^{\tau} \varepsilon_i \cdot x_i$$

for random $\varepsilon_i \in \{0, 1\}$.

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The squashed scheme from DGHV

The basic decryption $m \quad (c \bmod p) \bmod 2$ cannot be directly expressed as a boolean circuit of low depth.

Alternative decryption formula for $c = q \quad p + 2r + m$

- We have $q = bc/pe$ and $c = q + m \pmod 2$
- Therefore

$$m \quad [c]_2 \quad [bc \quad (1/p)e]_2$$

Idea (Gentry, DGHV). Secret-share $1/p$ as a sparse subset sum:

$$1/p = \sum_{i=1} s_i \quad y_i + \varepsilon$$

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- We have $q = \lfloor c/p \rceil$ and $c = q + m \pmod 2$
- Therefore

$$m \quad [c]_2 \quad [\lfloor c \quad (1/p) \rceil]_2$$

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with random public κ-bit numbers y_i, and sparse secret $s_i \ 2 \ f0,1g$.

Decryption becomes:

$$m \quad [c]_2 \quad \left[\left\lfloor \sum_{i=1} s_i \quad (y_i \quad c) \right\rceil \right]_2$$

Squashed decryption

Alternative equation

$$m \leftarrow [c]_2 \oplus [\lfloor c \cdot (1/p) \rceil]_2$$

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Decryption becomes:

$$m \leftarrow [c]_2 \oplus \left[\left\lfloor \sum_{i=1} s_i \cdot (y_i \cdot c) \right\rceil \right]_2$$

Gentry's Bootstrapping

The decryption circuit

- Can now be expressed as a polynomial of small degree d in the secret-key bits s_i, given the $z_i = c \quad y_i$.

$$m = C_{z_i}(s_1, \ldots, s \quad)$$

To refresh a ciphertext:

- Publish an encryption of the secret-key bits $\sigma_i = E_{pk}(s_i)$
- Homomorphically evaluate $m = C_{z_i}(s_1, \ldots, s \quad)$, using the encryptions $\sigma_i = E_{pk}(s_i)$
- We get $E_{pk}(m)$, that is a new ciphertext but possibly with less noise (a \recryption").
- The new noise has size $' \; d \quad \rho$ and is independent of the initial noise.

PK size and timings

Instance	λ	ρ	η	γ	pk size	Recrypt
Toy	42	27	1026	$150 \cdot 10^3$	77 KB	0.41 s
Small	52	41	1558	$830 \cdot 10^3$	437 KB	4.5 s
Medium	62	56	2128	$4.2 \cdot 10^6$	2.2 MB	51 s
Large	72	71	2698	$19 \cdot 10^6$	10.3 MB	11 min

Conclusion

Fully homomorphic encryption is a very active research area.

Main challenge: make FHE pratical !

Recent developments

- FHE without bootstrapping (modulus switching) [BGV11]
- Batch FHE [GHS12]
- Implementation with homomorphic evaluation of AES [GHS12]
- FHE based on matrix addition and multiplication [GSW13]
- HElib: FHE library of Halevi and Shoup [HS14]
- Faster Bootstrapping [AP13,AP14,DM15]